# Control Under Communication Constraints

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***Abstract*—There is an increasing interest in studying control systems employing multiple sensors and actuators that are geographically distributed. Communication is an important component of these distributed and networked control systems. Hence, there is a need to understand the interactions between the control components and the communication components of the distributed system. In this paper, we formulate a control problem with a communication channel connecting the sensor to the controller. Our task involves designing the channel encoder and channel decoder along with the controller to achieve different control objectives. We provide upper and lower bounds on the channel rate required to achieve these different control objectives. In many cases, these bounds are tight. In doing so, we characterize the "information complexity" of different control objectives.**

***Index Terms*—Communication, distributed systems, linear control, networked control.**

## I. Introduction

In this paper, we study linear, discrete time, control problems under communication constraints. Communication is an important component of distributed and networked control systems. Hence, there is a need to understand the fundamental relationship between how the control parts and the communication parts of the distributed system interact. A recent report on future research directions in control listed understanding control over communication networks as a major challenge for the controls field [10].

We examine observability and stabilizability under a communication constraint. The communication constraint is modeled as a discrete-time, noiseless, digital channel connecting the sensor to the controller. For each time step this channel is capable of transmitting $R$ bits without error. Our goal is to determine the minimum rate required on the channel to achieve our control objectives. In doing so, we characterize the "information complexity" of different control objectives. This paper presents a framework for treating communication issues in control problems.

This task entails specifying what the encoder, decoder, and controller, know and when they know it. This specification is called the information pattern [19]. We distinguish between two information patterns. In the first information pattern, we assume that the encoder has access to the control signals being applied to the plant. This may occur, for example, when the encoder is co-located with the plant. In the second information pattern, we assume that the encoder is geographically separated from the plant and hence observes the state, or output, but not the control signals being applied. We discuss the important role different information patterns can have on the control design.

We compute a lower bound on the rate required to achieve the different control objectives. This lower bound is independent of the information pattern in place and depends only on the plant. Specifically we show that a necessary condition on the rate for asymptotic observability and asymptotic stabilizability in a linear, discrete time, system is

$$R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$$

where the sum is over the eigenvalues of the $A$ matrix; see (1). This result relates the speed of the dynamics of the plant to the information rate of the channel. In the case where the encoder observes the control signals, the co-location case, we show that for any rate larger than this lower bound there exists an encoder, decoder, and controller that achieves the control objective. Hence, this bound is also sufficient. We then discuss upper bounds for systems where the encoder is geographically separated from the plant and, hence, does not have access to the control signals.

Typical communication channels are noisy and have delays. A complete understanding of the interaction between control and communication will need to use tools from both control theory and information theory. A necessary step in developing such a theory requires understanding the interaction between control and noiseless channels with bit rate constraints. This is the case examined in this paper. In general, one is interested in the relationship between control performance and communication rate. Understanding observability and stabilizability under communication constraints is an important initial step toward that larger goal.

In our companion paper, we extend these results [16] by showing the strong connection between control and the traditional information theoretic problems of source coding and channel coding. We present a general necessary condition for observability and stabilizability for a large class of noisy communication channels. Then, we study sufficiency conditions for Internet-like channels that suffer erasures.

Our problem formulation was inspired by [5]. There they formulated a linear quadratic control (LQC) problem under a rate-constrained channel. They discussed the interactions between coding, delay, and performance. The papers of Wong and Brockett were also influential [20], [21]. They provide conditions connecting the channel rate to the dynamics of the system to insure stabilizability of the system. We strengthen

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and generalize these results. Specifically, we present necessary and sufficient conditions on the channel for the multivariate case. Furthermore, we examine the role different information patterns have in determining the channel rate.

Elia and Mitter examined the stabilizability problem in the case where the encoder is time invariant [7]. They use a Lyapunov based synthesis scheme to design the underlying quantizers. Our work differs in that we allow for time-varying encoders. Liberzon and Brockett have also examined a Lyapunov based design with time-varying encoders [4]. They give upper bounds on the rate required to achieve stabilizability. A time-varying coding scheme is also presented in [13]. In this paper, we not only present bounds but also present conditions under which these bounds are tight.

Nair and Evans give rate conditions for stabilizability of an ARMA model [11]. In [12], they provide a rate condition similar to ours for insuring convergence. In contrast, our formulation allows us to analyze systems with process disturbances and different information patterns. Baillieul presents sufficient rate conditions for the multivariate case with a single input and an $A$ matrix with only real distinct eigenvalues [2], [3]. Here, we present a general framework for analyzing the multivariate state-space case with multiple inputs and arbitrary $A$ matrix. This framework allows us to prove necessary and sufficient rate conditions, treat different information patterns, and design encoding/decoding schemes that can treat output observations and are robust to process disturbances. Furthermore this framework lends itself to the treatment of control over more complicated communication channels as discussed in [16]. The work here can be found in [14] and has appeared in preliminary form in [15] and [17].

Here, are four observations that will motivate our analysis.

**Observation 1: Why feedback?**: If there is no uncertainty in the initial position, no uncertainty in the plant dynamics, and there are no process disturbances then one can achieve most control objectives using an open loop controller. A closed-loop controller for the same problem is often less complex to realize. Furthermore, a closed-loop controller can more robustly deal with the aforementioned uncertainties in initial position, plant dynamics and process disturbances. Thus, the point of feedback, if we bar complexity considerations, is to transmit from the plant to the controller information about the uncertainty in the state of the plant and the plant uncertainty itself that the controller does not know. The question then becomes what information is relevant and what communication scheme should be used to transmit that information.

**Observation 2: Full observation performance**: If the observation mechanism is instantaneous and the communication link is a lossless, infinite bandwidth, channel then we call the observation a full observation. We assume the control objective of interest is achievable under full observation. Clearly, if an objective cannot be achieved under full observation it cannot be achieved under the rate constrained observation. Equivalently, if a control objective can be achieved under a rate constrained observation then it can be achieved under full observation.

**Observation 3: Number of control sequences**: In a time horizon $T$ the decoder will receive one of at most $2^{TR}$ channel symbol sequences. If the encoder, decoder, and controller are also deterministic then the number of different possible control sequences in this time must be smaller than or equal to $2^{TR}$. Intuitively, then a control objective under rate $R$ can be achieved only if we can approximate well the control sequences for the full observation problem by one of only $2^{TR}$ control sequences. Thus, in terms of the underlying quantization problem one may think of quantization as living in the control sequence space.

**Observation 4: State estimation error**: The choice of channel rate influences the level of state estimation error in our observer. If the state estimation error increases with time in an unbounded fashion there will come a point when we can no longer satisfy the control objective. In this case, there is essentially no useful feedback. Thus, unless the control objective can be achieved via an open loop controller, i.e., a controller without access to any feedback, we cannot hope to achieve the control objective. Hence, in a rough sense, the state estimation error should grow at a slower rate than the state dynamics. We are interested in characterizing the largest tolerable level of state estimation error that still insures that the control objective is satisfied.

We conclude this introduction with a summary of the paper. In Section II, we formulate the problem. In Section III, we provide necessary conditions on the channel rate required to achieve observability and stabilizability. These bounds are independent of the information pattern chosen. Our next objective is to provide schemes that can achieve this bound. This depends heavily on the choice of information pattern. In Section IV, we discuss encoders with different information patterns. We conclude Section IV with an explicit quantizer construction and prove our main technical lemma.

In Section V, we treat encoder class one. This is the class where the encoder has access to the control signals. Here we show that the lower bounds provided in Section III can be attained. Specifically we provide a scheme that achieves the control objectives. In Section VI, we examine encoder class two. This encoder class has a more realistic information pattern. The rates, though, required to achieve the control objectives are larger than those in encoder class one. We conclude in Section VII.

## II. Problem Setup

We consider the following linear time-invariant system:

$$X_0 \in \Lambda_0, \qquad X_{t+1} = AX_t + BU_t \quad Y_t = CX_t \qquad \forall\, t \geq 0 \tag{1}$$

where $\{X_t\}$ is a $\mathbb{R}^d$-valued state process, $\{U_t\}$ is a $\mathbb{R}^m$-valued control process, and $\{Y_t\}$ is a $\mathbb{R}^l$-valued observation process. We have $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$, and $C \in \mathbb{R}^{l\times d}$. The initial position $X_0 \in \Lambda_0$ where $\Lambda_0 \subseteq \mathbb{R}^d$ is assumed to be an open set. If $C = I$, where $I$ is the identity matrix, then we have full-state observation at the encoder; see Fig. 1.

The communication channel is modeled as a noiseless digital channel that can transmit at each time step one of $2^R$ symbols denoted $\sigma \in \Sigma$, $|\Sigma| = 2^R$. Specifically at each time step the channel can transmit without error $R$ bits of information. Throughout this paper, log refers to logarithm base 2.

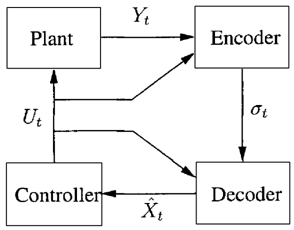


Fig. 1. System.

### A. Information Pattern

The control problems we look at involve the design of an encoder, decoder, and controller. Here, we specify the information pattern of each component. Let $X^t \doteq (X_0, \ldots, X_t)$.

*1) Encoder:* The encoder at time $t$ is a map

$$\mathcal{E}_t : \mathbb{R}^{l(t+1)} \times \Sigma^t \times \mathbb{R}^{mt} \to \Sigma \text{ taking}$$
$$(Y^t, \sigma^{t-1}, U^{t-1}) \mapsto \sigma_t.$$

In Section IV, we will discuss different restrictions on the available inputs to the encoder. The available inputs to the encoder are commonly called the *Information Pattern* of the encoder [19]. For example, the encoder may not have access to the past controls.

*2) Decoder:* The decoder at time $t$ is a map

$$\mathcal{D}_t : \Sigma^{t+1} \times \mathbb{R}^{mt} \to \mathbb{R}^d \text{ taking}$$
$$(\sigma^t, U^{t-1}) \mapsto \hat{X}_t.$$

The output of the decoder is an estimate of the state of the plant. We discuss how this estimate is computed in Section IV.

*3) Controller:* The controller at time $t$ is a map

$$\mathcal{C}_t : \mathbb{R}^d \to \mathbb{R}^m \text{ taking}$$
$$\hat{X}_t \mapsto U_t.$$

Note that we are assuming that the controller takes as input only the decoder's state estimate. Hence, we are assuming a separation structure between the decoder and the controller. We will show that for encoders in encoder class one, to be defined in Section IV, there is no loss of generality in making this separation assumption.

## III. Lower Bounds That are Independent of the Information Pattern

We now examine necessary conditions on the channel rate to insure observability and stabilizability. Note that the usual algebraic conditions for observability, e.g., certain Grammians having full rank, are still necessary but no longer sufficient. The additional necessary conditions take the form of lower bounds on the channel rate. These lower bounds will be universal in the sense that they hold independently of the actual encoder, decoder, and controller used. They hold independently of the information pattern in place.[1] In Section V, we show that there exists an information pattern, described in the definition of encoder class one, such that these lower bounds can be achieved. Thus, these lower bounds are tight.

[1]Note the analogy with Fano's inequality used in converse theorems in information theory [6]. Fano's inequality holds independently of the actual encoder and decoder used.

### A. Observability

The purpose of any good observer is to distinguish points in the state–space. In a time horizon of $T$, we have at most $2^{TR}$ possible symbols arriving at the decoder. Thus, at time $T$ we must be able to approximate the state by one of at most $2^{TR}$ points.

*Definition 3.1:* Let the *error* be $e_t = X_t - \hat{X}_t$ where $\hat{X}_t$ is the state estimate. System (1) is *asymptotically observable* if there exists an encoder and decoder such that the following hold for any control sequence $\{U_t\}$.

1) Stability: $\forall\, \epsilon > 0, \exists \delta(\epsilon)$ such that $\|X_0\|_2 \le \delta(\epsilon)$ implies $\|e_t\|_2 \le \epsilon, \forall\, t \ge 0$.
2) Uniform attractivity: $\forall\, \epsilon > 0, \forall\, \delta > 0 \exists T(\epsilon, \delta)$ such that $\|X_0\|_2 \le \delta$ implies $\|e_t\|_2 \le \epsilon, \forall\, t \ge T(\epsilon, \delta)$.

Point one states that the error cannot grow without bound for bounded $X_0$. The second point states that the error decreases to zero uniformly in $X_0$. Uniform attractivity is defined for all $\delta$. Thus our definition of asymptotic observability is global.

Traditional definitions of observability state that given enough time one can identify the initial condition exactly. Then once you know the initial condition and the controls, you can compute the state at any time in the future. In our case, at time $t$ we can only distinguish between $2^{tR}$ initial positions. Hence, there will be a certain amount of error in our state estimate of $X_0$. This error will propagate (due to the unstable modes of the system) in our estimate of any future $X_t$. It is for this reason that we introduce this definition of asymptotic observability.

Now, we are prepared to give a necessary condition on the rate required to achieve asymptotic observability. Let $\sum_{\lambda(A)}$ represent the sum over the eigenvalues of $A$. For a given set $\Omega \subseteq \mathbb{R}^d$ define the $\mathrm{diam}(\Omega) \doteq \sup_{x,y\in\Omega} \|x - y\|_2$.

*Proposition 3.1:* A necessary condition for system (1) to be asymptotically observable is $R \ge \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* Assume without loss of generality that the initial uncertainty contains the bounded set $\Omega_0 = \{X : \|X\|_2 \le L\} \subseteq \Lambda_0$. Note, possibly after a coordinate transformation, that the matrix $A$ can be written in the form

$$\begin{bmatrix} A_s & \\ & A_u \end{bmatrix}$$

where the $A_s$ block corresponds to the stable subspace (that subspace corresponding to the eigenvalues of $A$ that are strictly inside the unit circle) and the $A_u$ block corresponds to the marginally stable and unstable subspace (that subspace corresponding to the eigenvalues of $A$ that are either on the unit circle or outside the unit circle.) Let $\Pi_s$ represent the projection onto the stable subspace.

Fix an arbitrary control sequence $\{U_t\}$. Then, $X_t = A^t X_0 + \alpha_t$ where $\alpha_t = \sum_{j=0}^{t-1} A^{t-1-j} B U_j$. For any control sequence we have $\lim_{t\to\infty} \Pi_s(X_t - \alpha_t) = 0$. Thus, knowledge of the control signals alone is enough to estimate the projection of the state onto the stable subspace. Hence, without loss of generality, we can restrict our attention to $A$ matrices that contain only marginally stable and unstable eigenvalues.

The set of points that $X_t$ can take contains the following:

$$\Omega_t \doteq \{X : X = A^t X_0 + \alpha_t \text{ for some } X_0 \in \Omega_0\}.$$

If the system is asymptotically observable then it must be the case that $\forall\, \epsilon > 0$ and $\forall\, X_0 \in \Omega_0$ there is a $T(\epsilon, L)$ such that for $t \geq T(\epsilon, L)$ we have $\|e_t\|_2 \leq \epsilon$. In particular this must hold for $\epsilon < L$. A lower bound on the rate can be computed by counting the number of regions of diameter less than $2\epsilon$ it takes to cover $\Omega_t$ for $t \geq T(\epsilon, L)$.

If the diameter of a set $\Omega \subseteq \mathbb{R}^d$ is less than $2\epsilon$ then the volume of that set must be less than $K_d \epsilon^d$ (where $K_d$ is the constant in the formula for the volume of a sphere in $\mathbb{R}^d$.) Thus, to cover $\Omega_t$ by regions of diameter $2\epsilon$ we need at least

$$\begin{aligned} R &\geq \frac{1}{t}\log\frac{\text{volume}(\Omega_t)}{K_d\epsilon^d} \\ &\overset{a)}{=} \frac{1}{t}\log\frac{|\det(A^t)|\text{volume}(\Omega_0)}{K_d\epsilon^d} \\ &= \frac{1}{t}\log|\det(A^t)| + \frac{1}{t}\log\frac{K_d L^d}{K_d\epsilon^d} \\ &= \sum_{\lambda(A)} \log|\lambda(A)| + \frac{d}{t}\log\frac{L}{\epsilon} \end{aligned}$$

where a) follows from [1, Th. 10.38]. Note that since $\epsilon < L$ the term $(d/t)\log(L/\epsilon)$ is positive. Thus, $R \geq \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$ is a necessary condition for asymptotic observability. □

The following corollary shows that the rate bound in Proposition 3.1 continues to be necessary to insure bounded error in the case when there are additive process disturbances.

*Corollary 3.1:* Consider the case of bounded additive process disturbances: $X_{t+1} = AX_t + BU_t + W_t$ where $\|W_t\|_2 \leq D$. The rate $R \geq \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$ is necessary to insure that the state estimation error $\limsup_{t\to\infty} \|e_t\|_2$ is bounded.

*Proof:* As in Proposition 3.1, we can assume without loss of generality that the initial uncertainty set contains the bounded set $\Omega_0 = \{X : \|X\|_2 \leq L\} \subseteq \Lambda_0$. Also, we can partition the matrix $A$ into the $A_s$ and $A_u$ blocks. Let $\Pi_s$ represent the projection onto the stable subspace.

Fix an arbitrary control sequence $\{U_t\}$. Then, $X_t = A^t X_0 + \sum_{j=0}^{t-1} A^{t-1-j} BW_j + \alpha_t$ where $\alpha_t = \sum_{j=0}^{t-1} A^{t-1-j} BU_j$. For any control sequence, we have

$$\begin{aligned} &\limsup_{t\to\infty} \|\Pi_s(X_t - \alpha_t)\|_2 \\ &\quad = \limsup_{t\to\infty}\left(\left\|\Pi_s A^t\right\| L + \left\|\Pi_s \sum_{j=0}^{t-1} A^{t-1-j}\right\|_2 \|B\|_2 D\right) \\ &\quad < \infty. \end{aligned}$$

Because the initial position $X_0$ and the disturbances $W_t$ are bounded knowledge of the control signals alone is enough to insure that the error in the estimate of the projection of the state onto the stable subspace remains bounded. Thus, without loss of generality, we can restrict our attention to $A$ matrices that contain only marginally stable and unstable eigenvalues.

The set of points that $X_t$ can take contains the following:

$$\Omega_t \doteq \left\{X : X = A^t X_0 + \sum_{j=0}^{t-1} A^{t-1-j} BW_j + \alpha_t \quad \forall\, X_0 \in \Omega_0, \|W_t\|_2 \leq D\right\}.$$

Let $\Omega_t^{\text{zero}} = \{X : X = A^t X_0 + \alpha_t \text{ for some } X_0 \in \Omega_0\}$ be the set of points $X_t \in \Omega_t$ where all the disturbances are set to zero.

If there exists an encoder and decoder such that the estimation error is bounded then there must be a $\beta < \infty$ and a $T(L)$ such that $\|e_t\|_2 \leq \beta$ for $t \geq T(L)$. A lower bound on the rate can be computed by counting the number of regions of diameter less than $2\beta$ it takes to cover $\Omega_t$ for $t \geq T(L)$. Thus, we require a rate of at least

$$\begin{aligned} R &\geq \frac{1}{t}\log\frac{\text{volume}(\Omega_t)}{K_d\beta^d} \\ &\overset{a)}{\geq} \frac{1}{t}\log\frac{\text{volume}(\Omega_t^{\text{zero}})}{K_d\beta^d} \\ &\overset{b)}{\geq} \sum_{\lambda(A)} \log|\lambda(A)| + \frac{d}{t}\log\frac{L}{\beta} \end{aligned}$$

where a) follows because $\Omega_t^{\text{zero}} \subset \Omega_t$ and b) follows from Proposition 3.1. Note that $(d/t)\log(L/\beta)$ is bounded and becomes negligible as $t \to \infty$. □

Note that in the preceding proposition the necessary condition only depends on the uniform attractivity condition and not on the stability condition in the definition of asymptotic observability. Similarly, in the preceding corollary the necessary condition only depends on the boundedness condition.

### B. Stabilizability

In this section, we discuss stabilizability under a rate constraint. The lower bound uses a counting argument similar to that given in Proposition 3.1.

*Definition 3.2:* System (1) is *asymptotically stabilizable* if there exists an encoder, decoder, and controller such that the following holds.

1) Stability: $\forall\, \epsilon > 0$, $\exists \delta(\epsilon)$ such that $\|X_0\|_2 \leq \delta(\epsilon)$ implies $\|X_t\|_2 \leq \epsilon, \forall\, t \geq 0$.
2) Uniform attractivity: $\forall\, \epsilon > 0$, $\delta > 0 \exists T(\epsilon, \delta)$ such that $\|X_0\|_2 \leq \delta$ implies $\|X_t\|_2 \leq \epsilon, \forall\, t \geq T$.

Point one states that the state cannot grow unbounded for bounded $X_0$. The second point states that the state decreases to zero uniformly in $X_0$.

*Proposition 3.2:* Assume $(A, B)$ is a stabilizable pair. A necessary condition for (1) to be asymptotically stabilizable is $R \geq \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* As in Proposition 3.1, we can assume without loss of generality that the initial uncertainty set contains the bounded set $\Omega_0 = \{X : \|X\|_2 \leq L\} \subseteq \Lambda_0$ and that the matrix $A$ has only unstable and marginally stable eigenvalues.

For a given control sequence $U_0, U_1, \ldots, U_{t-1}$ we have $X_t = A^t X_0 + \sum_{i=0}^{t-1} A^{t-1-i} BU_i$. If the system is asymptotically stabilizable then it must be the case that $\forall\, \epsilon > 0$ and $\forall\, X_0 \in \Omega_0$ there exists a $T(\epsilon, L)$ such that $\forall\, t \geq T(\epsilon, L)$ we have

$\|X_t\|_2 \leq \epsilon$. In particular this must hold for $\epsilon < L$. For this value of $\epsilon$ define the sets $\Gamma$, parameterized by the control sequences $U_0, \ldots, U_{t-1}$, to be

$$\Gamma_{U_0^{t-1}} = \{X_0 : \|X_t\|_2 \leq \epsilon\}.$$

Note that $X_t$ depends linearly on $U_0, \ldots, U_{t-1}$, hence, all the $\Gamma$ sets are linear translations of each other. Thus, each $\Gamma$ set has the same volume: $\text{volume}(\Gamma) = |\det A^{-t}| K_d \epsilon^d$. A lower bound on the rate can be computed by counting how many $\Gamma$ sets it takes to cover $\Omega_0$. Now

$$\begin{aligned} R &\geq \frac{1}{t} \log \frac{\text{volume}(\Omega_0)}{\text{volume}(\Gamma)} \\ &= \frac{1}{t} \log \frac{K_d L^d}{|\det A^{-t}| K_d \epsilon^d} \\ &= \sum_{\lambda(A)} \log(|\lambda(A)|) + \frac{d}{t} \log\left(\frac{L}{\epsilon}\right). \end{aligned}$$

Since $\epsilon < L$ the term $(d/t)\log(L/\epsilon)$ is positive. We may conclude that $R \geq \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$ is a necessary condition for asymptotic stabilizability. □

## IV. Encoder Classes, Primitive Quantizers, and the Key Technical Lemma

In this section, we define two different encoder classes of interest, then we present the primitive quantizer, and we end with a statement and proof of the key technical lemma.

### A. Encoder Classes

Recall that the encoder at time $t$ is a map $\mathcal{E}_t$ that takes $(Y^t, \sigma^{t-1}, U^{t-1}) \mapsto \sigma_t$. In this case, the encoder knows the past outputs, past channel symbols, and past controls. In some cases, it is unreasonable to allow the encoder access to the past controls. For example, the encoder may be geographically separated from the plant.

At one extreme, one can consider an encoder with access to all the past information $(Y^t, \sigma^{t-1}, U^{t-1})$. At the other extreme, one can consider an encoder with access to only $Y_t$. There are many cases in between. We will provide an encoder, decoder, and controller for both of these two extreme cases. It will turn out that the encoder with access to the control signals has a lower rate requirement than the encoder without access to the control signals. These two cases shed light on the importance of the information pattern at the encoder for determining the channel rates required to achieve different control objectives.

**Encoder Class One**: In this class, the encoder at time $t$ is a map, $\mathcal{E}_t$, that takes $(Y^t, \sigma^{t-1}, U^{t-1}) \mapsto \sigma_t$. The decoder at time $t$ is a map, $\mathcal{D}_t$, that takes $(\sigma^t, U^{t-1}) \mapsto \hat{X}_t$. We assume that both the encoder and decoder have knowledge of the dynamics of the plant. We further assume that the encoder knows $\mathcal{D}_t$ and the decoder knows $\mathcal{E}_t$. We do not assume that the encoder or decoder knows the controller maps $\mathcal{C}_t$.

**Encoder Class Two**: In this class, the encoder at time $t$ is a map, $\mathcal{E}_t$, taking $Y_t \mapsto \sigma_t$. The encoder does not know the values of the control signals. We will assume, though, that it knows the control policy $\mathcal{C}_t$. The decoder at time $t$ is a map that takes $(\sigma^t, U^{t-1}) \mapsto \hat{X}_t$. Finally, we assume that the encoder knows $\mathcal{D}_t$ and the decoder knows $\mathcal{E}_t$.

### B. Equi-Memory

The encoder and decoder need to work together. The job of the decoder upon receiving the channel symbol $\sigma_t$ is to recover the quantization region the observation $Y_t$ fell into. To achieve this the decoder needs to know what the encoder operation is. Knowledge of the map $\mathcal{E}_t$ is not enough to insure this. To remedy this, we now introduce the notion of equi-memory [8].

We define a state for the encoder and decoder. This requires specifying the information pattern [19] of each encoder and decoder. The maximal amount of information the encoder can observe at time $t$ is $(Y^{t-1}, \sigma^{t-1}, U^{t-1}) \in \mathbb{R}^{lt} \times \Sigma^t \times \mathbb{R}^{mt}$. Let the information pattern of the encoder be $I_t \subset \mathbb{R}^{lt} \times \Sigma^t \times \mathbb{R}^{mt}$. For the decoder let $J_t = (\sigma^{t-1}, U^{t-1}) \in \Sigma^t \times \mathbb{R}^{mt}$. Finally, let $\Omega_{I_t,t}(\sigma) \doteq \{Y_t : \mathcal{E}_t(Y_t, I_t) = \sigma\}$.

*Definition 4.1:* An encoder/decoder pair are said to be *equi-memory* if the information $(J_t, \sigma_t)$ is sufficient to determine the set $\Omega_{I_t,t}(\sigma_t)$. Specifically there exists a map taking $(\sigma^t, U^{t-1}) \mapsto \Omega_{I_t,t}(\sigma_t)$.

In words, this definition states that the information in $J_t$ along with the quantization symbol $\sigma_t$ is sufficient to determine which particular quantization region the observation $Y_t$ fell into. We will assume throughout this paper that the encoder/decoder pairs we use are equi-memory. For encoder class one this means that $I_t = (Y^{t-1}, \sigma^{t-1}, U^{t-1})$ and hence the primitive quantizer should be chosen on the basis of the information $(\sigma^{t-1}, U^{t-1})$. For encoder class two this means that $I_t = \emptyset$ and hence the primitive quantizer should not depend on any of the control or channel signals.

### C. Primitive Quantizer

The encoders in both encoder classes will be restricted to apply the following primitive quantizer at each time step.

*Definition 4.2:* A *primitive quantizer* is a four-tuple $(c, \underline{R}, \underline{L}, \Phi)$ where $c \in \mathbb{R}^d$ represents the centroid, $\underline{R} = (R_1, \ldots, R_d)' \in \mathbb{R}^{d,+}$ represents the rate vector, $\underline{L} = (L_1, \ldots, L_d)' \in \mathbb{R}^{d,+}$ represents the dynamic range, and $\Phi$ is an invertible matrix that represents a coordinate transformation. This quantizer partitions the region

$$\Lambda = \left\{X \in \mathbb{R}^d : \Phi(X - c) \in \{[-L_1, L_1] \times \cdots \times [-L_d, L_d]\}\right\}$$

into boxes with side lengths $(2L_i/2^{R_i})$. Let $R = \sum_{i=1}^d R_i$. Each of the $2^R$ boxes is represented by an element $\sigma \in \Sigma$. Upon observing $X$ the $(c, \underline{R}, \underline{L}, \Phi)$-quantizer

1) subtracts off $c$;
2) applies the coordinate transformation $\Phi$;
3) determines which box it falls into (points that land on the boundary of more than one box are given the label of one of those boxes according to some fixed priority rule);
4) and then transmits the $\sigma$ representing that box.

If $X$ falls outside the region $\Lambda$ then the quantizer transmits a special symbol representing an overflow. Thus, we have $2^R + 1$ symbols. The set $\Lambda$ is called the *support* of the quantizer.

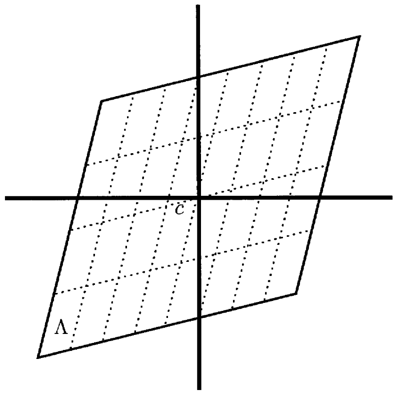


Fig. 2. Primitive quantizer.

Fig. 2 shows a two-dimensional primitive quantizer with $R_1 = 3$ and $R_2 = 2$. Here the dynamic range in both directions are equal: $L_1 = L_2 = 1$ and $c$ is the origin. The rate of this primitive quantizer is $R = 5$.

For both encoder class one and encoder class two, the encoding map $\mathcal{E}_t$ based on its information $I_t^*$ selects a $(c, \underline{R}, \underline{L}, \Phi)$-quantizer. Upon observing $Y_t$ it computes the appropriate $\sigma_t$ and transmits it across the channel. The decoder needs to know which quantizer was selected so that it may decode the received symbol $\sigma_t$ appropriately. This is assured by equi-memory. Specifically, the equi-memory condition forces the encoder and decoder to make decisions, in this case choose a primitive quantizer, based on the same information.

One may ask why we have chosen boxes instead of more general polytopes to partition $\Lambda$. Using general polytopes may lead to a lower rate than the rate one gets when restricting oneself to boxes. However, the analysis for the boxes case is much easier and they are simpler to implement in practice. Moreover, we will show that for certain cases encoders using primitive quantizers are sufficient to achieve the information theoretic lower bounds proved in Section III.

### D. Key Technical Lemma

The growth of the uncertainty in the state is determined by the $A$ matrix. Let us consider the uncontrolled system: $X_{t+1} = AX_t$. Assume $X_t \in \{[-L, L] \times \cdots \times [-L, L]\}$. We would like a way to compute a box that bounds the set that $X_{t+1}$ lives in given the box that $X_t$ lives in. In particular, we will characterize the growth of the box that upper bounds the state in terms of the dynamics of the state.

We first treat the simple case where the $A$ matrix has only real eigenvalues each with geometric multiplicity one. Let $\Phi$ diagonalize $A$: $\Phi A \Phi^{-1} = \Upsilon = \mathrm{diag}[\lambda_1, \ldots, \lambda_d]$. Let $Z_t = \Phi X_t$. Then $Z_{t+1} = \Phi X_{t+1} = \Phi A X_t = \Phi A \Phi^{-1} Z_t = \Upsilon Z_t$. If $Z_t \in \{[-L, L] \times \cdots \times [-L, L]\}$ then $Z_{t+1} \in \{[-|\lambda_1| L, |\lambda_1| L] \times \cdots \times [-|\lambda_d| L, |\lambda_d| L]\}$.

We now generalize this idea to general $A$ matrices. The construction we present requires the real Jordan canonical form. The following result can be found in [9, Sec. 6.4].

*Theorem 4.1:* For $A \in \mathbb{R}^{d \times d}$ there exists a real valued nonsingular matrix $\Phi$ and a real valued matrix $\Upsilon$ such that $\Phi A \Phi^{-1} = \Upsilon = \mathrm{diag}[J_1, \ldots, J_m]$. Where each $J_j$, $j = 1, \ldots, m$, is a Jordan block of dimension (geometric multiplicity) $d_j$. Clearly $d_1 + \cdots + d_m = d$. The Jordan block associated with a real eigenvalue $\lambda$ takes the form

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}.$$

The Jordan block associated with the complex conjugate pair of eigenvalues $\lambda = \rho(\cos\theta \pm i \sin\theta)$

$$\begin{bmatrix} D & I & & \\ & D & I & \\ & & \ddots & \\ & & & D \end{bmatrix}$$

where $D = \rho r(\theta)$ and $r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $(\rho \geq 0)$.

Define $H = \mathrm{diag}[H_1, \ldots, H_m]$ where each $H_j$ is associated with one of the Jordan blocks $J_j$. Specifically, $H_j = I$ if $J_j$ is the Jordan block with real eigenvalue $\lambda_j$ and $H_j = \mathrm{diag}[r(\theta)^{-1}, \ldots, r(\theta)^{-1}]$ if $J_j$ is the Jordan block associated with the complex conjugate eigenvalues $\rho(\cos\theta \pm i \sin\theta)$. If $A$ has all real eigenvalues then $H = I$. The following lemma, proved in Appendix, shows that $\Upsilon$ and any power of the matrix $H$ commute.

*Lemma 4.1:* $H^t \Upsilon H^{-t} = \Upsilon$.

As before, let $X_{t+1} = AX_t$ and let $Z_t = H^t \Phi X_t$. Note that if $A$ has real eigenvalues then $Z_t = \Phi X_t$. The $H$ is needed to undue the effects caused by the dynamics of the complex conjugate eigenvalue pairs. In general, $\Upsilon$ will not be an upper triangular matrix but the matrix $H\Upsilon$ will be an upper triangular matrix with real-valued eigenvalues. The magnitude of these eigenvalues are the same as the magnitude of the corresponding eigenvalues of $\Upsilon$. Now, $Z_{t+1} = H^{t+1} \Phi X_{t+1} = H^{t+1} \Phi A X_t = H^{t+1} \Phi A \Phi^{-1} H^{-t} Z_t = H^{t+1} \Upsilon H^{-t} Z_t = H \Upsilon Z_t$ where the last equality follows from Lemma 4.1.

We need a way to bound the growth of the operator $H\Upsilon$. This is an upper triangular matrix which can be written in the following block diagonal form $H\Upsilon = \mathrm{diag}[K_1, \ldots, K_m]$ where $K_j = J_j$ if $J_j$ is the Jordan block associated with a real eigenvalue and

$$K_j = H_j J_j = \begin{bmatrix} \rho I & r(\theta)^{-1} & & \\ & \rho I & r(\theta)^{-1} & \\ & & \ddots & \\ & & & \rho I \end{bmatrix}$$

if $J_j$ is the jordan block associated with a complex conjugate eigenvalue pair. The eigenvalues of the upper triangular matrix $K_j$ are all equal to $\rho$.

For each block $K_j$ associated with a real eigenvalue $\lambda_j$, define

$$\overline{K}_j \doteq \begin{bmatrix} |\lambda| & 1 & & \\ & |\lambda| & 1 & \\ & & \ddots & \\ & & & |\lambda| \end{bmatrix}.$$

For each block $K_j$ associated with a complex eigenvalue $\rho(\cos\theta + i\sin\theta)$, define

$$\overline{K}_j \doteq \begin{bmatrix} \rho I & O & & \\ & \rho I & O & \\ & & \ddots & \\ & & & \rho I \end{bmatrix}$$

where $O = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Finally, define $\overline{\Upsilon} \doteq \mathrm{diag}[\overline{K}_1, \ldots, \overline{K}_m]$.

*Lemma 4.2:* If $Z_t$ is in the box determined by $\underline{L}$ (i.e., $Z_t \in \{[-L_1, L_1] \times \cdots \times [-L_d, L_d]\}$) then $Z_{t+1}$ is in the box determined by $\overline{\Upsilon}\underline{L}$.

*Proof:* We know $Z_{t+1} = H\Upsilon Z_t$. By construction $\overline{\Upsilon}$ is a matrix whose $ij$th entry is greater than or equal to the absolute value of the $ij$th entry of $H\Upsilon$. Thus, if $Z_t$ is in the box determined by $\underline{L}$ then $Z_{t+1}$ will be in the box determined by $\overline{\Upsilon}\underline{L}$. □

We are now in a position to state the key technical lemma. This lemma shows how the growth in the box representing the uncertainty in the location of the state is determined by both the dynamics of the plant as well as the information transmitted across the channel. For a given rate vector $\underline{R} = (R_1, \ldots, R_d)'$ define:

$$F_{\underline{R}} \doteq \begin{bmatrix} 2^{-R_1} & & & \\ & 2^{-R_2} & & \\ & & \ddots & \\ & & & 2^{-R_d} \end{bmatrix}.$$

*Lemma 4.3:* If for all $i$ we have $R_i > \max\{0, \log|\lambda_i|\}$, then $\overline{\Upsilon}F_{\underline{R}}$ is stable. If there exists at least one $i$ such that $R_i < \max\{0, \log|\lambda_i|\}$, then $\overline{\Upsilon}F_{\underline{R}}$ is unstable.

*Proof:* The matrix $\overline{\Upsilon}$ is an upper triangular matrix whose eigenvalues have the same magnitude as the corresponding eigenvalues of the $A$ matrix. The matrix $\overline{\Upsilon}F_{\underline{R}}$ is an upper triangular matrix with the values $|\lambda_i|2^{-R_i}$ along its diagonal.

If for all $i$ the rate $R_i > \max\{0, \log|\lambda_i|\}$ then $0 \le |\lambda_i|2^{-R_i} < 1$ and, hence, all of the eigenvalues of $\overline{\Upsilon}F_{\underline{R}}$ will be stable. If $R_i < \max\{0, \log|\lambda_i|\}$ for at least one $i$ then $\overline{\Upsilon}F_{\underline{R}}$ will have at least one unstable eigenvalue. □

*Example:* Let $A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$, $\underline{R} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\underline{L}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The size of the box determined by $\underline{L}(t)$ will expand due to the dynamics of the system and then shrink due to the rate of information we transmit. Specifically, $\underline{L}(t+1) =$

$$\overline{\Upsilon}F_{\underline{R}}\underline{L}(t) = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2^{-2} & 0 \\ 0 & 2^{-3} \end{bmatrix}\underline{L}(t) = \begin{bmatrix} \frac{7}{8} \\ \frac{3}{8} \end{bmatrix}.$$

## V. Encoder Class One

In this section, we provide sufficient conditions on the rate for observability and stabilizability for systems with encoders restricted to encoder class one. In particular, we show that we can achieve the lower bounds proved in Section III. Furthermore, we show that the rate of convergence depends on the difference between the channel rate used and the lower bound.[2]

### A. Observability

Our first proposition treats asymptotic observability when the encoder observes the state ($Y_t = X_t$.) We follow this with two corollaries. The first provides a rate of convergence and the second shows that we can treat the case where the bound on the initial condition is unknown. We then generalize our basic observability result to systems with additive process disturbances and systems with output observations.

The results in this section are based on the following idea. At time $t$ the encoder first computes the decoder's estimate of the state. It can compute this estimate because it knows both the policy of the decoder and the control and channel signals available to the decoder. The encoder then computes the difference, the *innovation*, between the true value of the state and the decoder's estimate of the state. Next, the encoder chooses a primitive quantizer and quantizes this innovation. It then transmits to the decoder the appropriate channel symbol corresponding to this quantization value. The decoder, due to equi-memory, knows which primitive quantizer the encoder used and hence can decode this channel symbol and update its state estimate. In the following, we show that such a scheme leads to asymptotic observability for all rates satisfying the given rate bound.

There are many situations where the rate bounds presented in Section III will be noninteger. Clearly, we cannot send a noninteger number of bits at each time step. However, if we use a time-sharing scheme we can, on average, send a noninteger number of bits. We define the *average rate* of an encoding scheme to be $\limsup_{T\to\infty}(1/T)\sum_{t=0}^{T-1} R(t)$ where $R(t)$ is the number of bits transmitted at time $t$. We treat both the fixed rate and average rate cases in the following proposition. Let $\lceil x \rceil$ represent the smallest integer larger than or equal to $x$.

*Proposition 5.1:* For (1), encoders restricted to encoder class one $C = I$, and bounded initial set $\Lambda_0$

a) a sufficient condition on the *fixed* rate for asymptotic observability is $R > \sum_{\lambda(A)} \max\{0, \lceil \log|\lambda(A)| \rceil\}$;
b) a sufficient condition on the *average* rate for asymptotic observability is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* We will restrict our encoder and decoder to be equi-memory. We prove part a) first. Because $\Lambda_0$ is bounded we know there exists a constant $L$ such that $\Lambda_0 \subset \{X : \|X\|_2 \le L\}$. Let $\Phi$ diagonalize $A$ into real Jordan canonical form: $\Phi A \Phi^{-1} = \Upsilon$. For $X \in \Lambda_0$ we have $\|\Phi X\|_2 \le \|\Phi\|\|X\|_2 \le \|\Phi\|L$. At time zero choose a $(c(0), \underline{R}, \underline{L}(0), \Phi(0))$-quantizer where $c(0)$ is the origin and $\Phi(0) = \Phi$. Let $L_i(0) = \|\Phi\|L\ \forall\, i$ and choose the rate vector $\underline{R}$ such that each component $R_i$ is an integer and $R_i > \max\{0, \lceil \log|\lambda_i| \rceil\}$, $i = 1, \ldots, d$. Apply this quantizer to $X_0$ and transmit $\sigma_0$. Because the box determined by $\underline{L}(0)$ contains $\Lambda_0$ the channel symbol $\sigma_0$ will not be the overflow symbol.

[2]We can make an analogy with the rate distortion theorem in information theory. If the channel rate $C$ is less than the rate distortion rate $R$ then one cannot transmit information reliably (i.e., we cannot insure that the probability of decoding error converges to zero with time.) If $R < C$, then one can transmit reliably [6], [16].

At time $t$ let the state estimate, $\hat{X}_t$, be the centroid of the region defined by $\sigma_t$. Due to equi-memory both the encoder and decoder can determine this centroid. We update the quantizer parameters as follows.

1) The centroid of the $t+1$th quantizer is just the one-step ahead state prediction (the encoder observes the controls)

$$c(t+1) = A\hat{X}_t + BU_t.$$

2) The coordinate transformation evolves as

$$\Phi(t+1) = H^{t+1}\Phi = H\Phi(t).$$

As in Section IV let $Z_t = H^t \Phi X_t$ represent the state in the new coordinates. Then

$$Z_{t+1} = H\Upsilon Z_t + H^{t+1}\Phi B U_t.$$

3) The size of the dynamic range of the $t+1$th quantizer will evolve according to

$$\underline{L}(t+1) = \overline{\Upsilon} F_{\underline{R}} \underline{L}(t).$$

The $F_{\underline{R}}$ term captures the decrease in state estimation error at time $t$ due to applying the primitive quantizer while $\overline{\Upsilon}$ term captures the growth in the one-step ahead error due to the dynamics of the plant. By Lemma 4.3, $\overline{\Upsilon} F_{\underline{R}}$ is a stable matrix. Thus, the dimensions of the dynamic range are decreasing in time.

This completes our description of the encoder and decoder.

By construction, the state $X_t$ never leaves the range of the $t$th quantizer with support $\Lambda_t$, where

$$\Lambda_t = \{X \in \mathbb{R}^d : \Phi(t)(X - c(t)) \in \{[-L_1(t), L_1(t)] \times \cdots \times [-L_d(t), L_d(t)]\}\}.$$

Now, define

$$\Omega_t = \left\{X \in \mathbb{R}^d : \Phi(t)(X - \hat{X}_t) \in \left\{\left[-\frac{L_1(t)}{2^{R_1}}, \frac{L_1(t)}{2^{R_1}}\right] \times \cdots \times \left[-\frac{L_d(t)}{2^{R_d}}, \frac{L_d(t)}{2^{R_d}}\right]\right\}\right\}.$$

One can think of $\Lambda_t$ as containing the one-step ahead state prediction error for $X_t$ and $\Omega_t$ as containing the state estimation error for $X_t$ based on $\Lambda_t$ and the new information provided by the channel symbol $\sigma_t$. Now

$$\begin{aligned}
\|e_t\|_2 &\le \sup_{X \in \Omega_t} \|X - \hat{X}_t\|_2 \\
&= \sup_{X \in \Omega_t} \left\|\Phi(t)^{-1}\Phi(t)(X - \hat{X}_t)\right\|_2 \\
&\le \sup_{X \in \Omega_t} \|\Phi(t)^{-1}\|\|\Phi(t)(X - \hat{X}_t)\|_2 \\
&\le \|\Phi(t)^{-1}\|\|F_{\underline{R}}\underline{L}(t)\|_2 \\
&\le \|H^{-t}\|\|\Phi^{-1}\|\|F_{\underline{R}}\|\|(\overline{\Upsilon}F_{\underline{R}})^t\|\|\underline{L}(0)\|_2 \\
&\le \sqrt{d}L\|\Phi\|\|\Phi^{-1}\|\|F_{\underline{R}}\|\|(\overline{\Upsilon}F_{\underline{R}})^t\|.
\end{aligned}$$

Since $\|\Phi\|$, $\|\Phi^{-1}\|$, and $\|F_{\underline{R}}\|$ are all bounded we see that there exists a constant $\gamma$ such that $\|e_t\|_2 \le \gamma L\|(\overline{\Upsilon}F_{\underline{R}})^t\|$. Since $\overline{\Upsilon}F_{\underline{R}}$ is a stable matrix the error goes to zero as $t \to \infty$. Thus we have shown the uniform attractivity condition in the definition of asymptotic observability holds. Furthermore, for any $\epsilon > 0$ we can find an $L$ small enough so that for all $t \ge 0$ we have $\|e_t\|_2 \le \epsilon$. Thus the stability condition in the definition of asymptotic observability holds as well.

We will now prove part b) by specifying a time-sharing encoder and decoder scheme. In this case, the channel rate used at each time step can vary. Pick any $R_i > \max\{0, \log|\lambda_i|f\}$, $i = 1, \ldots, d$. There exist nonnegative integers $M$, $\alpha_i$, $\beta_i$, $i = 1, \ldots, d$ such that

$$R_i > \alpha_i + \frac{\beta_i}{M} > \max\{0, \log|\lambda_i|\}, \qquad i = 1, \ldots, d.$$

We will show that there exists an encoder and decoder scheme with average rate less than or equal to $\sum_{i=1}^d R_i$.

First, we need to specify a periodic schedule based on epochs of length $M$. At time $t$ apply a $(c(t), \underline{R}(t), \underline{L}(t), \Phi(t))$-quantizer where $c(t)$ and $\Phi(t)$ evolve as before. The sides of the dynamic range now evolve as $\underline{L}(t+1) = \overline{\Upsilon}F_{\underline{R}(t)}\underline{L}(t)$ where $\underline{R}(t) = [R_1(t), \ldots, R_d(t)]'$ is defined as follows $\forall\, i = 1, \ldots, d$:

$$R_i(t) = \begin{cases} \alpha_i + 1, & \text{if } t \bmod M \in \{0, 1, \ldots, \beta_i - 1\} \\ \alpha_i, & \text{if } t \bmod M \in \{\beta_i, \ldots, M-1\}. \end{cases}$$

For each $i = 1, \ldots, d$, we have $(1/T)\sum_{t=0}^{T-1} R_i(t) =$

$$\begin{aligned}
&\frac{1}{T}\left((T - T \bmod M)\left(\alpha_i + \frac{\beta_i}{M}\right) + \sum_{\tau=1}^{T \bmod M} R_i(\tau)\right) \\
&\quad = \left(\alpha_i + \frac{\beta_i}{M}\right) \\
&\qquad + \frac{1}{T}\left((-T \bmod M)\left(\alpha_i + \frac{\beta_i}{M}\right) + \sum_{\tau=1}^{T \bmod M} R_i(\tau)\right).
\end{aligned}$$

The second addend goes to zero as $T \to \infty$. Thus, for $T$ large enough we have: $\max\{0, \log|\lambda_i|\} < (1/T)\sum_{t=0}^{T-1} R_i(t) < R_i$. Hence, for $T$ large enough we have: $\sum_{i=1}^d \max\{0, \log|\lambda_i|\} < (1/T)\sum_{i=1}^d \sum_{t=0}^{T-1} R_i(t) < \sum_{i=1}^d R_i$.

Now, we need to show that under this time-sharing scheme the estimation error goes to zero as $t \to \infty$. As before

$$\|e_t\|_2 \le \|\Phi(t)^{-1}\|\|F_{\underline{R}(t)}\underline{L}(t)\|_2$$

where $\underline{L}(t) = \left(\prod_{\tau=1}^t \overline{\Upsilon}F_{\underline{R}(\tau)}\right)\underline{L}(0)$. The matrix $\prod_{\tau=1}^M \overline{\Upsilon}F_{\underline{R}(t)}$ is an upper triangular matrix. Using an argument similar to Lemma 4.3, we see that it is stable. Thus

$$\begin{aligned}
\|e_t\|_2 &\le \|\Phi(t)^{-1}\|\|F_{\underline{R}(t)}\underline{L}(t)\|_2 \\
&\le \|\Phi(t)^{-1}\|\|F_{\underline{R}(t)}\| \left\|\left(\prod_{\tau=1}^M \overline{\Upsilon}F_{\underline{R}(\tau)}\right)^{(t-(t \bmod M)/M)}\right\| \\
&\quad \times \left\|\prod_{\tau=1}^{(t \bmod M)} \overline{\Upsilon}F_{\underline{R}(\tau)}\right\| \|\underline{L}(0)\|_2
\end{aligned}$$

which goes to zero as $t \to \infty$. □

For the rest of this paper, the statement "a sufficient condition for property $X$ to hold is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$" will be taken to mean we are using the fixed rate scheme for integer rates and the average rate scheme for noninteger rates.

*Corollary 5.1:* For both the fixed rate and average rate encoding schemes described in Proposition 5.1 the rate of convergence is $\|e_t\|_2 \leq \kappa t^{(d_{\max}-1)} 2^{-t(\min_i(R_i - \log|\lambda_i(A)|))}$ where $\kappa$ is a constant independent of $t$.

*Proof:* We treat the fixed rate case. A similar argument can be used for the average rate case. The decay of $\|(\overline{\Upsilon} F_{\underline{R}})^t\|$ is determined by the largest eigenvalue of $\overline{\Upsilon} F_{\underline{R}}$. The largest eigenvalue is given by $\max_i 2^{(-R_i+\log\overline{|\lambda_i(A)|})}$. There exists a constant $\eta$ such that $\|(\overline{\Upsilon} F_{\underline{R}})^t\| \leq \eta t^{(d_{\max}-1)} 2^{-t(\min_i(R_i-\log|\lambda_i(A)|))}$ where $d_{\max}$ is the multiplicity of the largest eigenvalue. Thus

$$\begin{aligned}\|e_t\|_2 &\leq \sqrt{d} L \|\Phi\| \|\Phi(t)^{-1}\| \|F_{\underline{R}}\| \|(\overline{\Upsilon} F_{\underline{R}})^t\| \\ &\leq \kappa t^{(d_{\max}-1)} 2^{-t(\min_i(R_i-\log|\lambda_i(A)|))}\end{aligned}$$

for a constant $\kappa$ independent of $t$ (Recall $\|\Phi(t)^{-1}\| \leq \|H^{-t}\| \|\Phi^{-1}\| \leq \|\Phi^{-1}\|$.) □

Next, we treat the case when $\Lambda_0$ is bounded but the encoder does not *a priori* know that bound.

*Corollary 5.2:* For (1), encoders restricted to encoder class one, and $C = I$ a sufficient condition on the rate for asymptotic observability is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* For the case when a bound on the initial uncertainty, $\Lambda_0$, is unknown one must first "capture" the state in the quantizer domain. Let $\Phi$ be as in Proposition 5.1. At time zero apply the primitive quantizer $(c(0), \underline{R}(0), \underline{L}(0), \Phi)$ where $c(0)$ is the origin, $\Phi(0) = \Phi$, and $L_i(0) = L \,\forall\, i$ for some fixed $L$. If upon observing $X_t$ the quantizer transmits an overflow symbol then update the quantizer as follows: $c(t+1) = Ac(t) + BU_t$, $\Phi(t) = H^t\Phi$, and $\underline{L}(t+1) = 2\overline{\Upsilon}\underline{L}(t)$. Since $\underline{L}(t)$ is growing at a rate faster than the dynamics of the state process eventually the quantizer will capture the state. At this point, proceed as we did in Proposition 5.1. (Strictly speaking, we are using one extra symbol for the overflow message. However, as $t \to \infty$ the overhead due to these overflow messages is amortized out.) □

We now consider the case of bounded additive process disturbances

$$X_{t+1} = AX_t + BU_t + W_t \quad Y_t = X_t, \qquad t \geq 0 \tag{2}$$

where $\|W_t\|_2 \leq D$. In Corollary 3.1, we provided a necessary condition to insure bounded error. The following proposition provides a related sufficient condition.

*Proposition 5.2:* For (2), encoders restricted to encoder class one, and bounded initial set $\Lambda_0$ the rate $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$ is sufficient to insure $\limsup_{t\to\infty} \|e_t\|_2$ is bounded.

*Proof:* First, we prove the fixed rate case. We follow the proof in Proposition 5.1. The state dynamics in the new coordinates are: $Z_{t+1} = H\Upsilon Z_t + \Phi(t+1)BU_t + \Phi(t+1)W_t$. We update the $\underline{L}(t)$ as follows:

$$\underline{L}(t+1) = \overline{\Upsilon} F_{\underline{R}} \underline{L}(t) + D\|\Phi(t+1)\| \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Here, we expand the size of the dynamic range of the quantizer to take into account the dynamics of the plant, the information transmitted across the channel, and the bounded process disturbances. For shorthand, denote the second addend by $h(t)$. Then, we can write $\underline{L}(t) = (\overline{\Upsilon} F_{\underline{R}})^t \underline{L}(0) + \sum_{j=0}^{t-1} (\overline{\Upsilon} F_{\underline{R}})^{t-1-j} h(j)$.

As in Proposition 5.1, we have

$$\begin{aligned}\|e_t\|_2 &\leq \|\Phi(t)^{-1}\| \|F_{\underline{R}} \underline{L}(t)\|_2 \\ &\leq \|\Phi(t)^{-1}\| \|F_{\underline{R}}\| \\ &\quad \times \left( \left\|(\overline{\Upsilon} F_{\underline{R}})^t \underline{L}(0)\right\|_2 + \left\| \sum_{j=0}^{t-1} (\overline{\Upsilon} F_{\underline{R}})^{t-1-j} h(j) \right\|_2 \right) \\ &\leq \sqrt{d} \|\Phi(t)^{-1}\| \|F_{\underline{R}}\| \\ &\quad \times \left( \left\|(\overline{\Upsilon} F_{\underline{R}})^t \underline{L}(0)\right\|_2 \right. \\ &\qquad \left. + \sum_{j=0}^{t-1} D \left\|(\overline{\Upsilon} F_{\underline{R}})^{t-1-j}\right\| \|H^{j+1}\| \|\Phi\| \right) \\ &\leq \sqrt{d} \|\Phi(t)^{-1}\| \|F_{\underline{R}}\| \\ &\quad \times \left( \left\|(\overline{\Upsilon} F_{\underline{R}})^t \underline{L}(0)\right\|_2 \right. \\ &\qquad \left. + D\|\Phi\| \sum_{j=0}^{t-1} \|(\overline{\Upsilon} F_{\underline{R}})^{t-1-j}\| \right).\end{aligned}$$

The matrix $(\overline{\Upsilon} F_{\underline{R}})$ is stable. Hence, the first term goes to zero as $t \to \infty$. For the second term, note that there exists a constant $\kappa$ such that $\|(\overline{\Upsilon} F_{\underline{R}})^t\| \leq \kappa t^{(d_{\max}-1)} 2^{-t(\min_i(R_i-\log|\lambda_i(A)|))}$. Thus, the series is summable. Hence, there exists an $\eta$ such that $\lim_{t\to\infty} \sum_{j=0}^{t-1} \|(\overline{\Upsilon} F_{\underline{R}})^{t-1-j}\| \leq (\eta/1 - 2^{-\min_i(R_i - \log|\lambda_i|)})$. Therefore $\lim_{t\to\infty} \|e_t\| \leq (\eta\sqrt{d}D/1 - 2^{-\min_i(R_i-\log\lambda_i)}) \|\Phi(t)^{-1}\| \|\Phi\| \|F_{\underline{R}}\|$. A similar argument can be used for the average rate case. □

Due to the noise term $W_t$ there will always be a nonzero state estimation error. But if the channel rate goes to infinity then we see that $\|F_{\underline{R}}\| = \max_i 2^{-R_i} \to 0$ and, as we expect, the bound on the estimation error in the previous proposition goes to zero.

*Example:* Consider the scalar system: $X_{t+1} = aX_t + bU_t + W_t$. In this case, the limit of the error is bounded as $\lim_{t\to\infty} |e_t| \leq (D/2^R - |a|)$.

Now, we consider (1) with general observation equation $Y_t = CX_t$. Assume that the pair $(A, C)$ is observable. We will need the following technical lemma which is proved in Appendix.

*Lemma 5.1:* Let $A$ be a stable matrix. Let $B_t$ be a set of matrices such that $\|B_t\| \leq K < \infty$ and the $\lim_{t\to\infty} \|B_t\| = 0$. Let $S_t = \sum_{i=0}^{t-1} A^{t-1-i} B_i$. Then $\lim_{t\to\infty} \|S_t\| = 0$.

*Proposition 5.3:* For (1), encoders restricted to encoder class one, and bounded initial set $\Lambda_0$ a sufficient condition for asymptotic observability is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* At time $t$, the encoder receives $Y^t, U^{t-1}$. Assume that the encoder has access to an observer that produces a state estimate $\bar{X}_t$. Specifically the encoder applies a Luenberger observer: $\bar{X}_t = A\bar{X}_{t-1} + BU_{t-1} + M(Y_{t-1} - C\bar{X}_{t-1})$ where $M$ is chosen so that $A - MC$ is stable. The estimation error of this observer is $\bar{e}_t = X_t - \bar{X}_t = (A - MC)\bar{e}_{t-1}$. Thus

$\bar{e}_t = (A-MC)^t\bar{e}_0$ and $\|\bar{e}_t\| \leq \|(A-MC)^t\|\|\bar{e}_0\| \leq \eta\alpha^t\|\bar{e}_0\|$ for some constant $\eta$ and $0 \leq \alpha < 1$. The observer's initial state error $\|\bar{e}_0\|$ is bounded because $\Lambda_0$ is bounded. Hence, as $t \to \infty$ the observer's state error $\bar{e}_t$ goes to zero.

We will construct an encoder and decoder that asymptotically observes the observer state $\bar{X}_t$. The observer's estimation error goes to zero, hence, the encoder and decoder will asymptotically observe $X_t$. The observer's state estimate evolves as $\bar{X}_t = A\bar{X}_{t-1} + BU_{t-1} + MC\bar{e}_{t-1}$. We will treat the $MC\bar{e}_{t-1}$ term as a process disturbance.

First, we treat the fixed rate scheme. As in Proposition 5.2, we update $\underline{L}(t)$ as follows:

$$\underline{L}(t+1) = \overline{\Upsilon}F_{\underline{R}}\underline{L}(t) + c\alpha^t\|MC\|\|\bar{e}_0\|\|\Phi(t+1)\| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

where $c = (\eta/\alpha)$. Denote the second addend by $\alpha^t h(t)$. Then, $\underline{L}(t) = (\overline{\Upsilon}F_{\underline{R}})^t\underline{L}(0) + \sum_{j=0}^{t-1}(\overline{\Upsilon}F_{\underline{R}})^{t-1-j}\alpha^j h(j)$. Since $\overline{\Upsilon}F_{\underline{R}}$ is stable the first term goes to zero. Because $\lim_{t\to\infty}\alpha^t h(t) = 0$ ($h(t)$ is uniformly bounded in $t$) Lemma 5.1 implies that the second term goes to zero. Thus, as $t \to \infty$ the error in the decoder's estimate of the encoder's observer state converges to zero. Thus, the rate condition is sufficient for asymptotic observability. A time-sharing argument can be used for the average rate case. □

The general prescription for observability in encoder class one is to transmit a finer and finer description of the zero control input response state trajectory. If there are no disturbances this is equivalent to successively refining the initial position. Specifically, if we allow the encoder "infinite" memory then it need only transmit a finer and finer description of $X_0$. However, assuming that the encoder can have in memory a perfect description of $X_0$ for all time $t$ is unrealistic. Furthermore, it is not robust to disturbances. For this reason, we have proposed a recursive structure for the encoders in encoder class one. As we have shown, this recursive structure can be used for systems with process disturbances and output observations.

### B. Stabilizability

For encoder class one, we can combine the properties of asymptotic observability and full state feedback stability to get output feedback stability. Assume that the pair $(A, B)$ is stabilizable and the pair $(A, C)$ is observable.

*Proposition 5.4:* For (1), encoder restricted to encoder class one, and the initial set $\Lambda_0$ bounded a sufficient condition for asymptotic stabilizability is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* Let $K$ be a stabilizing controller, i.e., $A + BK$ is stable. Apply the certainty equivalent controller $U_t = K\hat{X}_t$ where $\hat{X}_t$ is the decoder's state estimate. We will show that under this control law the system is stable. Let $e_t = X_t - \hat{X}_t$. Then $X_t = (A+BK)^t X_0 - \sum_{j=0}^{t-1}(A+BK)^{t-1-j}BKe_j$. By Proposition 5.3 we can asymptotically observe the state. Thus, the $\lim_{t\to\infty}\|e_t\|_2 = 0$. Since $A + BK$ is stable and $\Lambda_0$ is bounded the first addend in the aforementioned equation goes to zero. By Lemma 5.1 so does the second. Hence $\lim_{t\to\infty}\|X_t\|_2 = 0$. Thus, we have shown the uniform attractivity condition in the definition of asymptotic stabilizability holds. Furthermore, for any $\epsilon > 0$ we can find an $L$ small enough so that for all $t \geq 0$ we have $\|X_t\|_2 \leq \epsilon$. Thus, the stability condition in the definition of asymptotic stabilizability holds as well. □

This result is related to a general result of Vidyasagar that states if a system is state feedback stabilizable and output observable then it is output feedback stabilizable [18]. Furthermore, the certainty equivalent controller applied to the state estimate is a stabilizing controller. The difference here is that due to the encoder and decoder the "observation equation" will depend on the past states and controls.

We can treat the case when $\Lambda_0$ is bounded but the encoder does not *a priori* know that bound.

*Corollary 5.3:* For system (1), encoder restricted to encoder class one, and $C = I$ a sufficient condition for asymptotic stabilizability is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$.

*Proof:* Apply the zero control until the encoder, using the technique in Corollary 5.2, "captures" the state. Then, proceed as in Proposition 5.4. □

*Example:* Consider the scalar system $X_{t+1} = aX_t + bU_t$, $a > 1$, $|X_0| \leq L$. Choose a controller $k$ such that $|a + bk| < 1$. Under full state feedback, the magnitude of the state is strictly decreasing to the origin: $|X_t| = |a+bk|^t|X_0|$. Under a rate $R > \log a$ and the scheme proposed previously, we see $|X_t| \leq |a+bk|^t|X_0| + \sum_{j=0}^{t-1}|a+bk|^{t-1-j}|bk|(1/2^{Rj})L$. There can exist trajectories that initially are not strictly decreasing to the origin. One can consider this the price of learning the state under a rate constraint; see [7].

## VI. Encoder Class Two

In this section, we examine observability and stabilizability for encoders restricted to encoder class two. For this encoder class we assume that the encoder knows the control law though not the actual control signals.

### A. Observability

For encoder class one, we showed there exist encoders that can asymptotically observe the state. Furthermore, because the encoder observed the controls, asymptotic observability held independently of the control signals chosen. Here, we will show that the condition of observability for encoders in encoder class two depends on the control signals chosen.

*Proposition 6.1:* For (1), with unstable $A$, encoder restricted to encoder class two, initial set $\Lambda_0$, $C = I$, and controls set to zero there is no finite rate encoder in encoder class two such that the estimation error is bounded.

*Proof:* As in Proposition 3.1, we can assume without loss of generality that the initial uncertainty set contains the bounded set $\Omega_0 = \{X : \|X\|_2 \leq L\} \subseteq \Lambda_0$ and that the $A$ matrix contains only marginally stable and unstable eigenvalues. The set of points that $X_t$ can take contains the following set $\Omega_t \doteq \{X : X = A^t X_0, X_0 \in \Omega_0\}$.

If there exists an encoder and decoder in encoder class two such that the estimation error is bounded then there must be a $\beta < \infty$ and a $T(L)$ such that $\|e_t\|_2 \leq \beta$ for $t \geq T(L)$. A lower bound on the rate can be computed by counting the number of regions of diameter less than $2\beta$ it takes to cover $\Omega_t$ for $t \geq T(L)$.

At any time $t_0 > T(L)$, we know that any $X_{t_0} \in \Omega_{t_0}$ is a realizable state. To insure an error of size $\beta$ at time $t_0$ we would need to transmit a total of at least

$$\log \frac{\text{volume}(\Omega_{t_0})}{K_d \beta^d} = t_0 \sum_{\lambda(A)} \log|\lambda(A)| + d \log \frac{L}{\beta}$$

bits over the $t_0$ time steps. Thus, for the first $t_0$ steps we would need to transmit bits at a rate of at least $R \geq \sum_{\lambda(A)} \log|\lambda(A)| + (d/t_0)\log(L/\beta)$. By the same argument, at time $t_1 = t_0 + N$ we would need to transmit a total of at least $t_1 \sum_{\lambda(A)} \log|\lambda(A)| + (d/t_0)\log(L/\beta)$ bits over $t_1$ time steps. Encoders in encoder class two are memoryless hence the encoder at any given time cannot depend on any of the previous encodings of the state. Thus, we cannot use any of the bits transmitted during the first $t_0$ time steps to help aid the reconstruction of $X_{t_1}$. Therefore over the time interval from $t_0$ to $t_1$ we would need to transmit bits at a rate of at least $R \geq (1/N)\left(t_1 \sum_{\lambda(A)} \log|\lambda(A)| + d\log(L/\beta)\right)$. For fixed $N$ we see that the right-hand side grows to infinity as $t_0$, and hence $t_1$, goes to infinity. Thus, there does not exist a finite rate scheme that will insure bounded error. □

In summary, the encoder at time $t$ cannot depend on the previous encodings and hence cannot compute the innovation with respect to the previous encodings. Thus there can be no finite rate encoder in encoder class two such that the estimation error is bounded.

This result may seem like bad news. However, in most cases we are interested in observability when the controls being applied are in the closed loop. We will now show that we can achieve asymptotic observability and as a by-product of asymptotic stabilizability.

### B. Stabilizability

Assume $(A, B)$ is a stabilizable pair. Because the encoders in encoder class two do not observe the control signals they need to operate with the closed loop dynamics of $A + BK$ and not the open loop dynamics of $A$. We have assumed that the encoder knows the control law $K$. Computing the minimal rate is difficult for this class of encoders because in general the rate will depend on the particular controller policy $K$.

*Proposition 6.2:* For (1), encoder restricted to encoder class two, $C = I$, and bounded initial set $\Lambda_0$ there exists a finite rate such that the system is asymptotically stabilizable.

*Proof:* Choose $K$ such that $A + BK$ is stable. Let $\Phi$ diagonalize $(A + BK)$ into real Jordan canonical form: $\Phi(A + BK)\Phi^{-1} = \Upsilon$. Assume $\Lambda_0 \subset \{X : \|X\|_2 \leq L\}$. For $X \in \Lambda_0$ we have $\|\Phi X\|_2 \leq \|\Phi\|\|X\|_2 \leq \|\Phi\|L$. At time zero choose a $(c(0), \underline{R}, \underline{L}(0), \Phi(0))$-quantizer where $c(0)$ is the origin and $\Phi(0) = \Phi$. Let $L_i(0) = \|\Phi\|L \; \forall\, i$. The rate vector $\underline{R}$ will be determined shortly. Apply this quantizer to $X_0$ and transmit $\sigma_0$. Note that $\sigma_0$ will not be the overflow symbol.

The encoder does not have access to the controls or the past channel symbols. Thus, it can only evolve according to a schedule based on the control law. The state evolves as: $X_{t+1} = (A + BK)X_t - BKe_t$. Update the quantizer parameters as follows: $c(t) = 0$ for all time and $\Phi(t+1) = H\Phi(t)$. Where $H$ is defined as in Section IV but, in this case, it is with respect to the matrix $A + BK$. Then, the state in the new coordinates evolves as: $Z_{t+1} = H\Upsilon Z_t - \Phi(t+1)BKe_t$ where $e_t = X_t - \hat{X}_t$. Let $f_t = Z_t - \hat{Z}_t = \Phi(t)e_t$. Then

$$Z_{t+1} = H\Upsilon Z_t - \Phi(t+1)BK\Phi^{-1}(t)f_t.$$

Observe that $Z_t$ is in the box determined by $\underline{L}(t)$ and the error $f_t$ is in the box determined by $F_{\underline{R}}\underline{L}(t)$. Let the boundary lengths of the dynamic range evolve as

$$\underline{L}(t+1) = \left\{\overline{\Upsilon} + \|\Phi(t+1)BK\Phi^{-1}(t)\|F_{\underline{R}}\right\}\underline{L}(t).$$

The operator $\overline{\Upsilon}$ is stable because $\Upsilon$ is stable. We can then find a rate vector $\underline{R}$ with components large enough so that $\overline{\Upsilon} + \|\Phi(t+1)BK\Phi^{-1}(t)\|F_{\underline{R}}$ is uniformly stable in $t$. Thus, $\lim_{t\to\infty} \underline{L}(t) = 0$ and the system is asymptotically stabilizable. □

*Example:* Consider the scalar system $X_{t+1} = aX_t + bU_t$. Let $k$ be such that $|a + bk| < 1$. Then

$$L(t+1) = \left(|a + bk| + \frac{|bk|}{2^R}\right)L(t).$$

Letting $R > \max\{0, \log(|bk|/1 - |a + bk|)\}$ is sufficient to ensure asymptotic stabilizability. If $a + bk = 0$ then the rate bound becomes $R > \max\{0, \log|a|\}$.

For both encoder class one and two, our strategy has been to keep track of a region in which the state lies. The size and location of this region are recursively updated. One might wonder how these regions, determined by the dynamic range and support of the quantizer, are related to the Lyapunov level sets used in Lyapunov theory. In Lyapunov theory one finds a suitable Lyapunov function $V$ such that the value of $V$ decreases along trajectories [4], [7]. For a finite channel rate it is impossible to insure that $V$ decreases at every time step. This is because there is always a region around the origin that is not under the influence of any control.

## VII. Conclusion

In this paper, we formulated a discrete time, linear systems control problem with a noiseless digital communication link. We discussed the role of information patterns and control policy knowledge in this context.

We first provided lower bounds on the rates required to achieve asymptotic observability and asymptotic stabilizability. These bounds hold independently of the information pattern chosen. To compute upper bounds we explicitly described the encoder, decoder, and controller schemes. We characterized two different encoder structures based on whether the encoder observed the control signals or not. Under the added structural assumptions of equi-memory and use of a primitive quantizer we showed that encoders in encoder class one can achieve these lower bounds. For encoders in encoder class a weaker result was provided.

## APPENDIX

Let $A \in \mathbb{R}^{d\times d}$. Then, by theorem 4.1 $A$ has a real Jordan canonical form.

*Lemma 4.1:* $H^t \Upsilon H^{-t} = \Upsilon$.

*Proof:* $H^t \Upsilon H^{-t}$ is the product of three block diagonal matrices. Thus, we need only check that the result holds for each of the blocks. The blocks come in two types: those associated with real eigenvalues and those associated with complex eigenvalues. For the real eigenvalue case $H_j$ is identity. Thus, clearly $I^t J_j I^{-t} = J_j$. Let us examine the complex conjugate eigenvalue case

$$H_j^t J_j H_j^{-t} = \mathrm{diag}[r(\theta)^{-t}] \begin{bmatrix} \rho r(\theta) & I & & \\ & \rho r(\theta) & I & \\ & & \ddots & \\ & & & \rho r(\theta) \end{bmatrix} \times \mathrm{diag}[r(\theta)^t] = \begin{bmatrix} \rho r(\theta) & I & & \\ & \rho r(\theta) & I & \\ & & \ddots & \\ & & & \rho r(\theta) \end{bmatrix} = J_j.$$

□

*Lemma 5.1:* Let $A$ be a stable matrix. Let $B_t$ be a set of matrices such that $\|B_t\| \le K$ and the limit $\lim_{t\to\infty} \|B_t\| = 0$. Let $S_t = \sum_{i=0}^{t-1} A^{t-1-i} B_i$. Then, $\lim_{t\to\infty} \|S_t\| = 0$.

*Proof:* Since $A$ is stable there exists $c \ge 0$ and $0 \le \lambda < 1$ such that $\|A^t\| \le c\lambda^t$. For all $\epsilon > 0$ there exists a $T(\epsilon)$ such that $\|B_t\| \le \epsilon, \forall\, t \ge T(\epsilon)$. Let $t > T(\epsilon)$. Then

$$\begin{aligned} \left\| \sum_{j=0}^{t-1} A^{t-j-1} B_j \right\| &\le \sum_{j=0}^{t-1} \|A^{t-j-1}\| \|B_j\| \\ &\le c \sum_{j=0}^{t-1} \lambda^{t-j-1} \|B_j\| \\ &\le c \left\{ \lambda^{t-T(\epsilon)-1} \sum_{j=0}^{T(\epsilon)} \lambda^{T(\epsilon)-j} K \right. \\ &\qquad \left. + \sum_{j=T(\epsilon)+1}^{t-1} \lambda^{t-j-1} \epsilon \right\} \\ &\le \frac{c}{1-\lambda} \left\{ \lambda^{t-T(\epsilon)-1} K + \epsilon \right\}. \end{aligned}$$

Now, $\epsilon > 0$ can be chosen arbitrarily small and $t > T(\epsilon)$ can be chosen arbitrarily large. Hence, the bound can be made arbitrarily small. □

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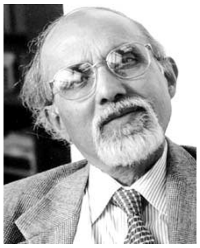

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